DADSS Lecture - January 27, 2000

IRR - Internal Rate of Return

-IRR is the "break-even" interest rate.

-How is IRR calculated?

-First, look at calculating NPV:

$$NPV = CF_0 + \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + \cdots + \frac{CF_N}{(1+r)^N}$$

-For some series of cash flows (CF) and some interest or discount rate (*r*), we arrive at the net present value of the cash flow stream. NPV is *some* number.

-For IRR, we want to solve for the *r* that will make the NPV = 0.

-Consider a shorter cash flow stream:

Year 0: -$10
Year 1: $12

-What is the NPV if the interest rate is 10%?

$$NPV = -10 + \frac{12}{(1+r)^1} = -10 + \frac{12}{1.1} = 0.91$$

-Now...what would the IRR be? We want NPV = 0:

$$0 = -10 + \frac{12}{(1+r)^1}$$

$$10 = \frac{12}{(1+r)^1}$$

$$10(1+r) = 12$$

$$1+r = \frac{12}{10}$$

$$r = 1.2 - 1 = 20\%$$

-The internal rate of return is 20%. That is, this project will break even (have a positive or zero NPV) until the discount rate exceeds 20%.

-But there's a complication. What happens with larger cash flow streams? The IRR is the root of the discounted cash flow equation, which is a large polynomial.

2 periods (highest power = 1)	Easy
3 periods (highest power = 2)	Quadratic Formula
4 periods (highest power = 3)	Cubic Roots...
⋮	
6 periods (highest power = 5)	Barely possible
7+ periods	Need Computer

-Excel uses an algorithm (Newton-Raphson) to quickly find the correct IRR.

-Another complication: Descartes' Rule of Signs. The number of roots (or zeroes) of a polynomial is equal to the number of sign changes or one.

Cash flow stream #1:

-10 +2 +2 +5 +5 +10
^^^^^^^

Only 1 sign change.

Cash flow stream #2:

-10 +2 +2 -5 +5 -10 +15

5 sign changes! So 5 possible IRRs.

A Perpetuity

-What is the value of an infinite stream of cash flows?

-At 10%, $100 is worth...

...in 1 year $\frac{\$100}{1+0.1}=\$90.91$

...in 2 years $\frac{\$100}{(1+0.1)^2}=\$82.64$

...in 10 years $\frac{\$100}{(1+0.1)^{10}}=\$38.55$

...in 100 years $\frac{\$100}{(1+0.1)^{100}}=\$0.007$

...in ∞ years $\lim_{n\to\infty}\frac{\$100}{(1+0.1)^n}=\frac{\$100}{\infty}=\$0$

-Combining all these together as with NPV:

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{CF_i}{(1+r)^i}=\frac{CF}{r}$$

-The value of a perpetuity is the cash flow divided by the interest rate or discount rate.

Annuities

-Suppose you were going to receive $100 per year for 10 years and the interest rate was 10%. What is that worth today?

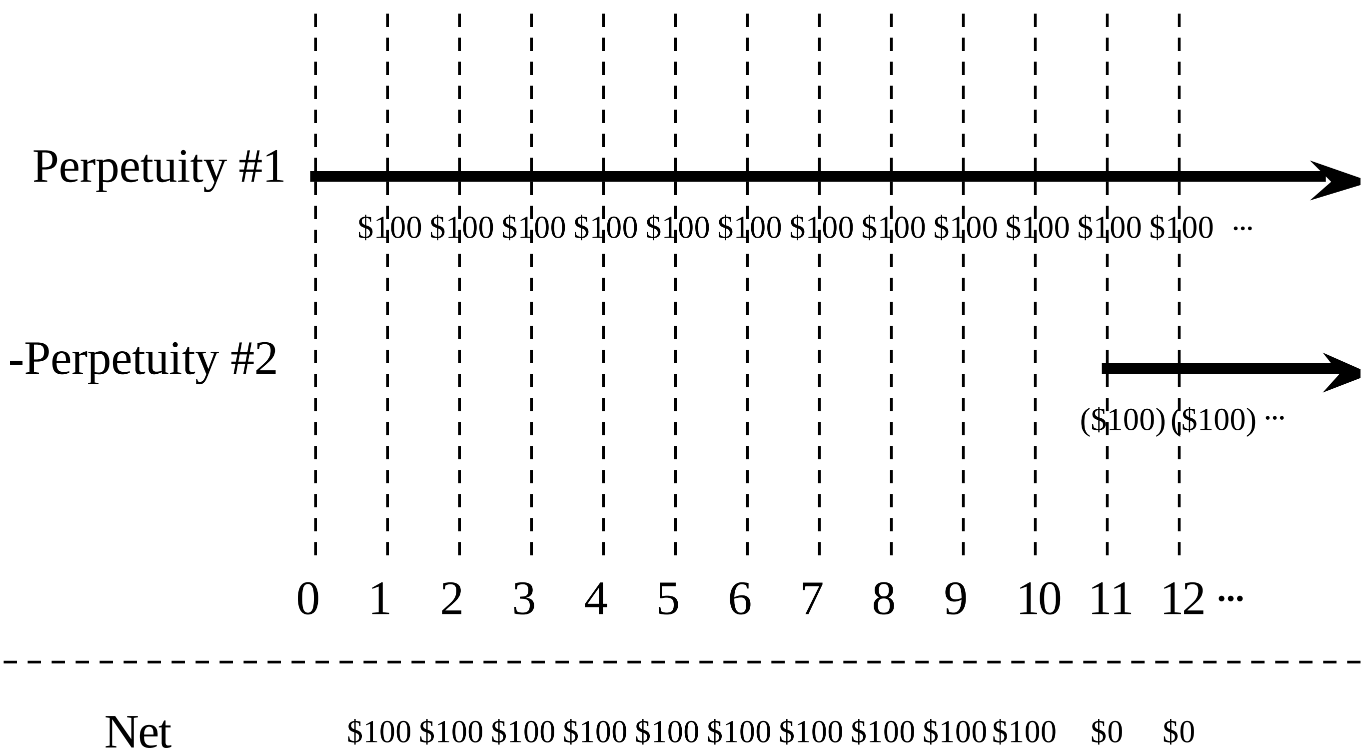
-A perpetuity would give you the value of such a cash flow stream *forever*, but we just want the value for 10 years.

-How could we use what we know about perpetuities and NPV to calculate the value of this *annuity*.

Annuities as Perpetuities...

-It turns out that the value of an annuity is equal to the difference between two perpetuities.

-Consider the following two cash flow streams which are effectively equivalent (when combined) to a 10 year annuity:



-So, the value of the first perpetuity is $\frac{\$100}{10\%} = \$1{,}000.00$.

The value of the second perpetuity is also $1,000 *but in the 10th year!* We need to bring this back to the 0th year with all the other cash flows. This, however, should now be easy for you:

$$\frac{\$1{,}000}{(1+0.1)^{10}} = \$385.54$$

-So, the value of the annuity is:

$1,000.00 - $385.54 = $614.46

-Mathematically,

$$\begin{aligned}\text{Annuity} &= \text{Long Perpetuity} - \text{Short Perpetuity}\\ &= \frac{CF}{r} - \frac{CF}{r}\left(\frac{1}{(1+r)^N}\right)\\ &= \frac{CF}{r}\left[1 - \frac{1}{(1+r)^N}\right]\end{aligned}$$

Nominal and Effective Interest Rates

-Suppose you put $100 in a savings account that earned 5% per year. At the end of the year, you'd have your original $100 and $5 in interest.

-However, if the bank paid interest more frequently, you'd actually end up with more money as your interest began to earn interest. Suppose that the bank paid interest twice a year (2.5% each time. For the second six months, you would be earning money not on $100, but on $102.50.

-At the end of the year, you'd have:

$$\$100\left(1+\frac{0.05}{2}\right)^2=\$100(1.050625)=\$105.06.$$

-What if the bank paid interest daily (daily compounding)?

$$\$100\left(1+\frac{0.05}{365}\right)^{365}=\$100\,(1.0512675)=\$105.13$$

-Finally, what if the bank paid interest every instant (continuous compounding)? Is this even practical? No, but it's easy to use...

How do we get it?

-Let's look at the general formula for compounding and converting between nominal (the 5%) interest rates and effective (the 5.12675%) interest rates:

$$\text{Effective} = \left(1 + \frac{\text{Nominal Interest Rate}}{\text{Num of Compounding Periods}}\right)^{\text{\# of Comp. Periods}}$$

-As the number of compounding periods goes to infinity, we can use a mathematical result:

$$\lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^{k} = e$$

-Do you see the similarity? In the continuous case, the effective interest rate is equal to:

$$r_{\text{effective}} = e^{r_{\text{nominal}}} - 1$$

-Thus, if the bank paid interest continuously at 5%, at the end of the year you would have

$$\$100e^{0.05} = \$100\,(1.0512711) = \$105.13$$