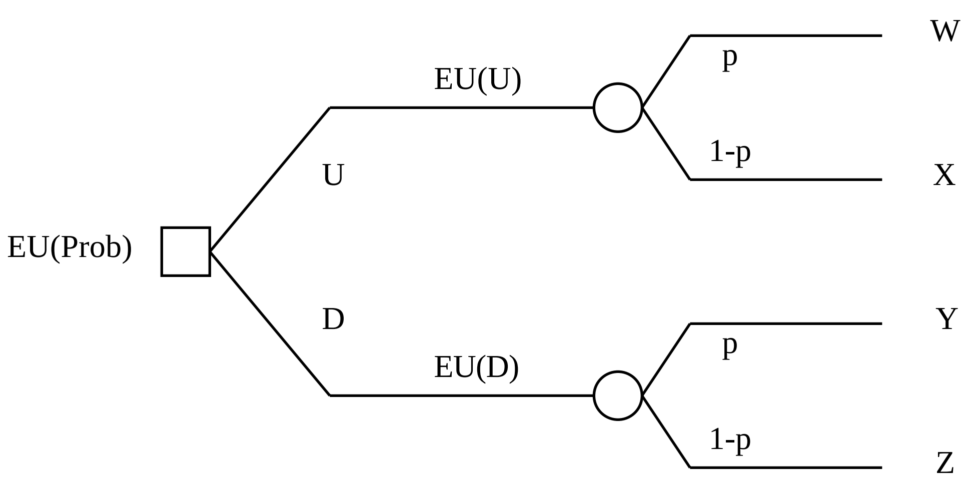
A Proof on Certainty Equivalence Under Exponential Utility

Proposition: When an agent's utility can be given in terms of a generalized exponential function, the expected value of perfect information is equal to the simple difference between the certainty equivalents of the base no information case and the perfect information case: $EVPI = CE^{\star}_{PI} - CE_B$, where B refers to the base "no information" case and PI refers to the case with perfect information.

Proof: Let $u(x) = e^x$ and $W > Z > Y > X$. Let $EU(\cdot)$ indicate the expected utility operator. The generic decision problem is:



Normally, the certainty equivalent is equal to x, where $EU(\text{Prob}) \triangleq e^x$. Note that the notation $\triangleq$ or 'set equal' indicates the preliminary step in solving for the variable in question. Thus, it indicates a *defined* equality rather than a *natural* equality. Then:

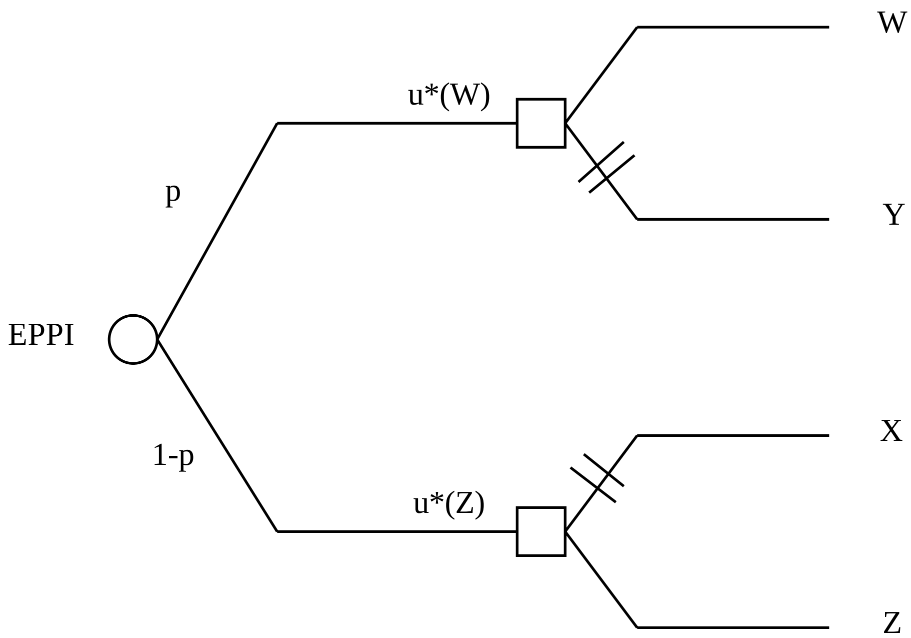
$$CE_B = x = \ln\left[EU(\text{Prob})\right]$$

and

$$EU(\text{Prob}) = \max\left\{EU(\text{U}), EU(\text{D})\right\}$$

$$EU(\text{U}) = pu(W) + (1-p)u(X)$$
$$EU(\text{D}) = pu(Y) + (1-p)u(Z)$$

With perfect information, of cost *c*, we have:



$$EPPI = pu^{\star}(W) + (1-p)u^{\star}(Z)$$

where $u^{\star}(\cdot)$ reflects the incorporation of the cost of information, *c*.

Then, by the strong monotonicity of $u(\cdot)$ and the ordering of outcomes, EVPI (or *c*) can be obtained thus:

$$EU(\text{Prob}) \triangleq pu(W-c) + (1-p)u(Z-c)$$

This is because it was assumed that $W > Z > Y > X$. Solve for *c*:

$$\begin{aligned} EU(\text{Prob}) &= pe^{W-c} + (1-p)e^{Z-c} \\ &= pe^{W}e^{-c} + e^{Z}e^{-c} - pe^{Z}e^{-c} \\ &= e^{-c}\left(pe^{W} + (1-p)e^{Z}\right) \end{aligned}$$

$$e^{-c} = \frac{EU(\text{Prob})}{pe^{W} + (1-p)e^{Z}}$$

$$c = -\ln\left(\frac{EU(\text{Prob})}{pe^{W} + (1-p)e^{Z}}\right)$$

But, the proof statement indicates that $CE^{\star}_{PI}$ should be used. This is the certainty equivalent for perfect information *without* taking into account the cost of the perfect information. This case is simple - we invert the utility function to solve for *x* (which represents the cost for this case):

$$
\begin{aligned}
EPPI &\overset{\text{set}}{=} e^{x} \\
x &= \ln(EPPI) \\
&= \ln[pu(W) + (1-p)u(Z)] \\
&= \ln[pe^{W} + (1-p)e^{Z}] \\
&= \ln(pe^{W} + e^{Z} - pe^{Z}) \\
CE^{\star}_{PI} \Rightarrow x &= \ln(p(e^{W} - e^{Z}) + e^{Z})
\end{aligned}
$$

The claim then, is that $CE^{\star}_{PI} - CE_{B} = EVPI$. By making use of a well-known property of logarithms, $\log\left(\frac{x}{y}\right) = \log x - \log y$, and through some straightforward algebra, we obtain the desired result:

$$\ln(p(e^{W} - e^{Z}) + e^{Z}) - \ln[EU(\text{Prob})] = -\ln\left(\frac{EU(\text{Prob})}{pe^{W} + (1-p)e^{Z}}\right) \qquad \blacksquare$$

Proof for a Generalized Exponential Utility Function

Let $u(x) = \alpha + \beta e^{\gamma x}$ and $\beta, \gamma \neq 0$. By using the decision trees given previously, we can calculate:

$$EU(U) = pu(W) + (1-p)u(X)$$
$$EU(D) = pu(Y) + (1-p)u(Z)$$

$$EU(B) = \max\{EU(U), EU(D)\}$$

Therefore, $CE_{B} = x = \frac{1}{\gamma}\ln\left(\frac{EU - \alpha}{\beta}\right)$.

Now, consider perfect information, of cost *c*:

$$EU(PI) = pu^{\star}(W,c) + (1-p)u^{\star}(Z-c)$$

We must consider two cases in calculating the EVPI:

Case I: The Regular Method of Calculating the EVPI

Find the value of *c* such that $EU(B) = pu(W-c) + (1-p)u(Z-c)$.

$$
\begin{aligned}
EU(B) &= p\left[\alpha+\beta e^{\gamma(W-c)}\right]+(1-p)\left[\alpha+\beta e^{\gamma(Z-c)}\right]\\
EU(B)-\alpha &= e^{-\gamma c}\left[\beta p e^{\gamma W}-\beta e^{\gamma Z}-\beta p e^{\gamma Z}\right]\\
e^{-\gamma c} &= \frac{EU(B)-\alpha}{\beta p e^{\gamma W}-\beta e^{\gamma Z}-\beta p e^{\gamma Z}}\\
\therefore \quad c &= -\frac{1}{\gamma}\ln\left[\frac{EU(B)-\alpha}{\beta p e^{\gamma W}-\beta e^{\gamma Z}-\beta p e^{\gamma Z}}\right]
\end{aligned}
$$

Case II: Calculating the "Costless" Perfect Information EVPI

$$
\begin{aligned}
EU(PI) &= \alpha+\beta e^{\gamma x}\\
x &= \frac{1}{\gamma}\ln\left[\frac{EU(PI)-\alpha}{\beta}\right]\\
&= \frac{1}{\gamma}\ln\left[\frac{\beta p e^{\gamma W}+\beta e^{\gamma Z}-\beta p e^{\gamma Z}}{\beta}\right]\\
&= \frac{1}{\gamma}\ln\left[p e^{\gamma W}+(1-p)e^{\gamma Z}\right]=CE_{PI}
\end{aligned}
$$

The claim is that $CE_{PI}-CE_{B}=EVPI=c$. This is equivalent to proving that:

$$
\begin{aligned}
\frac{1}{\gamma}\ln\left[p e^{\gamma W}+(1-p)e^{\gamma Z}\right]-\frac{1}{\gamma}\ln\left[\frac{EU(B)-\alpha}{\beta}\right] &=\\
&= \frac{1}{\gamma}\ln\left[\frac{\beta\left(p e^{\gamma W}-(1-p)e^{\gamma Z}\right)}{EU(B)-\alpha}\right]\\
&= -\frac{1}{\gamma}\ln\left[\frac{EU(B)-\alpha}{\beta p e^{\gamma W}-\beta e^{\gamma Z}-\beta p e^{\gamma Z}}\right]
\end{aligned}
$$

This statement, which proves the claim, is possible due to two properties of logarithms:

$$
\begin{aligned}
\log x-\log y &= \log\left(\frac{x}{y}\right)\\
\log x &= -\log\left(\frac{1}{x}\right)
\end{aligned}
\qquad \blacksquare
$$

David Rode
Decision Analysis and Decision Support Systems
Department of Social and Decision Sciences
Carnegie Mellon University
September 12, 1998