

# **Introduction to Linear Programming**

Algorithmic and Geometric Foundations of Optimization

# Optimization and Linear Programming

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- “Mathematical programming” is a class of methods for solving problems which ask you to optimize: *to maximize, minimize, to find the best, the worst, the most, the least, the fastest, the shortest, etc.* Linear programming is the simplest form, suitable only for problems with linear objective functions and constraints.
- Two of the Basic Linear Programming Problems:
  - The Product Mix Problem
    - What quantities of products should be produced to maximize profit?
    - In economics, this is the optimal allocation of scarce resources.
  - The Blending Problem
    - What combination of ingredients minimizes cost?
    - For example, creating industrial chemical compounds.

## Some Optimization Basics

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•Unconstrained:  $\max_x f(x) = x$

Easy:  $\infty$

•Unconstrained:  $\max_x f(x) = x(1-x)$

Still easy:  $\frac{df(x)}{dx} = 1 - 2x = 0 \quad \therefore x^* = \frac{1}{2}$

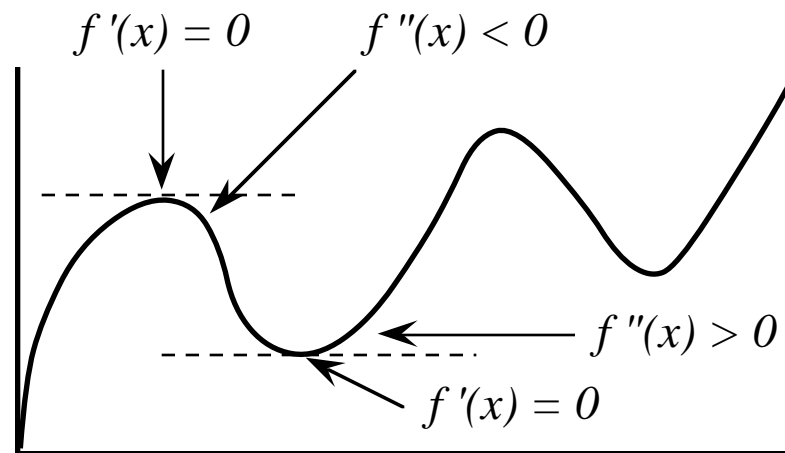
•In general, what are we doing?

-Set the derivative to zero and solve.

-Why?

- Necessary and Sufficient Conditions for an optimum over single-peaked functions:

	Necessity	Sufficiency
Maximum	$\frac{df(x)}{dx} = 0$	$\frac{d^2f(x)}{dx} < 0$
Minimum	$\frac{df(x)}{dx} = 0$	$\frac{d^2f(x)}{dx} > 0$



## Some Complications

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- What about more complicated functions?

$$\max_{x, y, z} f(x, y, z) = x(1 - y) + y(1 - x) + z(1 - x - y)$$

- Calculus still works, but is cumbersome to do by hand. We turn to computers and automate the process instead.

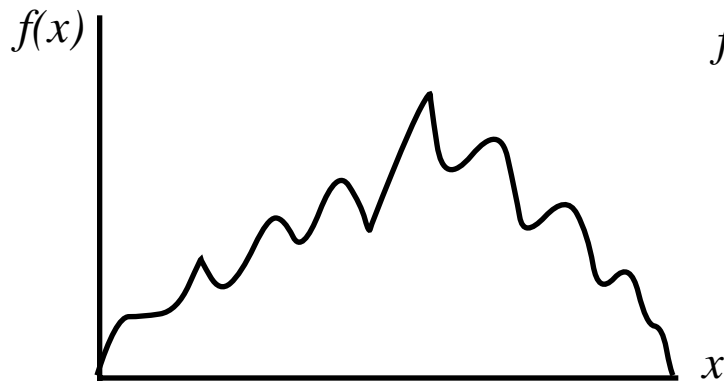
- What about constraints?

$$\max_x f(x) = x(1 - x), \text{ subject to } x \leq \frac{1}{3}$$

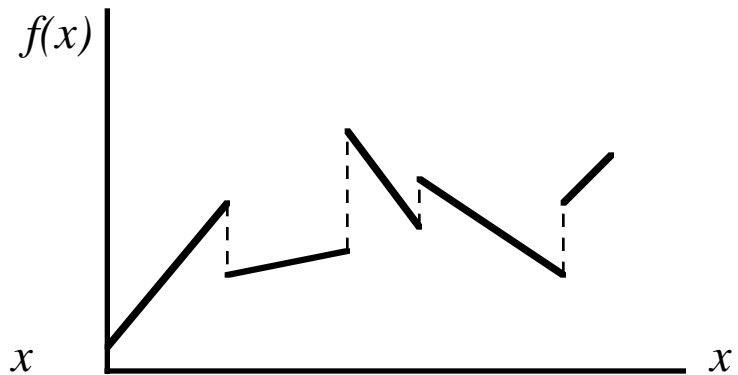
- Again, calculus works (use the Lagrangian) for small problems, but needs automation for larger problems.

# More Complications...

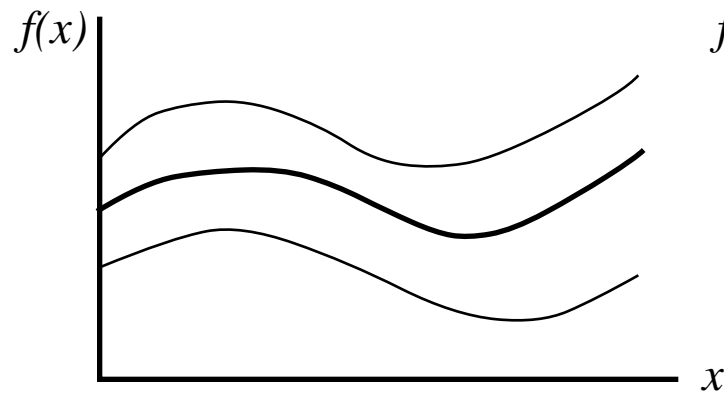
Multiple Optima



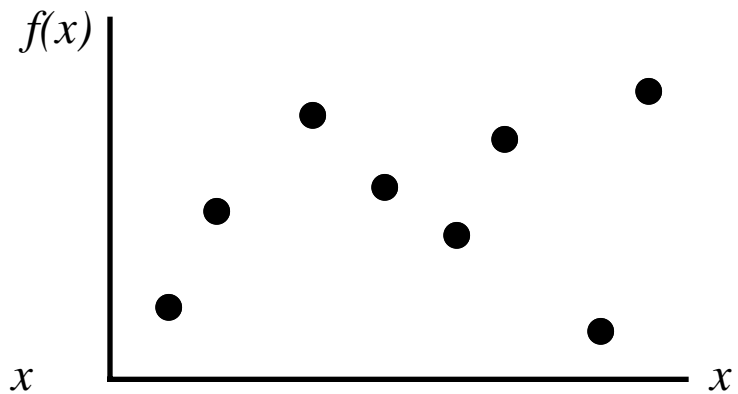
Discontinuous Functions



Uncertainty



Integer Restrictions



## The Need for Algorithmic Methods

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- How can we automate searching for optima?
- Linear programming (LP) is one such method.
- LP is easy and fast, but it has some important limitations:

### Limitations/Requirements

- (1) Decisions are made under certainty
- (2) Objective function and constraints must be linear
- (3) Additivity
- (4) Complete Independence
- (5) Infinite Divisibility

### Alternatives

- (1) Sometimes expectations can be used, but generally stochastic programming is required
- (2) NLP (tough!). No guaranteed global optimizer other than enumeration! GA, GP, SA, heuristic search, tabu, adaptive grid refinement, active sets
- (3) NLP
- (4) NLP
- (5) Integer programming, branch & bound

## LP Structure

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- Assuming the requirements for LP have been met, what does a LP look like?
- Decision Variables
  - Quantities to be allocated
  - Number of units to be produced
  - Time intervals, unit quantities, proportions...
- Objective Function
  - What is being maximized or minimized?
  - *Thought: How do we reduce the world to a set of linear equations?*
- Constraints
  - (In)equalities specifying limits on the unbounded optimization of a function.
  - If  $\Pi = P \times Q$ , why not just set  $Q$  to  $\infty$ ?



## Setting up a LP Problem: Product Mix

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Fischbeck Electronics makes two models of television sets,  $A$  and  $B$ . Fischbeck Electronics' profit,  $\Pi_A$ , on  $A$  is \$300 per set and its profit on  $B$ ,  $\Pi_B$ , is \$250 per set. Thus, the problem facing Fischbeck Electronics is:

$$\max_{x_A, x_B} \bar{\Pi} = \Pi_A x_A + \Pi_B x_B$$

Notice that *unconstrained* optimization would cause them to produce only  $A$ ! But, Fischbeck Electronics faces several constraints:

- (1) **A Labor Constraint:** They can't use more than 40 hours of labor per day (5 employees working regular shifts).
- (2) **A Manufacturing Constraint:** They only have 45 hours of machine time available (9 machines available for 5 hours each).
- (3) **A Marketing Constraint:** They can't sell more than 12 units of set  $A$  per day.

## Identifying the Problem Variables

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- Decision Variables

- What can we vary?

- $x_A$  and  $x_B$  are the only variables we have direct control over.

- The Objective Function

- What are we maximizing (minimizing)?

- Profit! (Cost!)

- Thought: Max Profit = Min Cost = Max -Cost = ...*

- How is it determined?

- $\bar{\Pi} = \Pi_A x_A + \Pi_B x_B$ , but we know that  $\Pi_A = 300$  and  $\Pi_B = 250$ .

- So, the objective function is  $\bar{\Pi} = 300x_A + 250x_B$ .

## Identifying the Constraints

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• 3 primary constraints *plus* feasibility ( $x_i \geq 0 \ \forall i$ ):

(1) Labor: *A* needs 2 hours of labor. *B* needs only 1 hour.  
Total labor is equal to the number of units produced times the labor required per unit.

$$2x_A + x_B \leq 40$$

(2) Manufacturing: *A* needs 1 hour of machine time. *B* needs 3 hours.

$$x_A + 3x_B \leq 45$$

(3) Marketing: Can't see more than 12 units of *A* per day.

$$x_A \leq 12$$

## The Entire LP and Possible Implications

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$$\max_{x_A, x_B} \bar{\Pi} = 300x_A + 250x_B$$

$$\text{such that: } 2x_A + x_B \leq 40$$

$$x_A + 3x_B \leq 45$$

$$x_A \leq 12$$

$$x_A \geq 0$$

$$x_B \geq 0$$

- What if  $x_A$  or  $x_B = 7.4235$ ? What does it mean to produce 0.4235 TV sets?
- Can 0.6689 people be employed?

## Solving Linear Programming Problems

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- Often, setting up LPs is the hardest part. This is because the actual “solving” is usually done by computer.
- How do computers do it?
  - We will use a graphical technique to illustrate the intuition behind the algorithmic methods used in optimization software packages.
- For any LP, there exists an infinite number of possible solutions.
- Problem spaces may be vast. Many models have tens of thousands of variables and constraints.
- How can we search? How can we limit the number of possibilities we have to try?

## Algorithmic Solutions

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- The Simplex Algorithm (and variants)

- Created by George Dantzig in 1947 for the Air Force Office of Scientific Research

- “If the LP problem has an optimal solution, it will be found in a finite number of iterations.” (Note: this isn’t always very comforting)

- The simplex algorithm finds the solution by making iterated improvements from an initial position (solution).

- The Ellipsoid Algorithms

- The best known example is Karmarkar’s Algorithm (created at AT&T Bell Labs)

- Runs in linear time

- Faster than simplex for *very* large problems

- We won’t worry about Karmarkar...

## What about the Solver?

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• Do not use Excel or Solver as a black box! Know what's going on underneath the surface. Remember: you will always get *an* answer back from Solver. However, knowing whether or not that answer is relevant requires understanding what Solver is doing. Know its limitations!

• ***Linear programs***: Standard simplex

• ***Quadratic programs***: Modified simplex (QP is technically NLP, but results specific to quadratic forms can be exploited to allow for significant simplification.

• ***Nonlinear Programs***: Generalized Reduced Gradient Method. NOTE: Unless you check *Assume linear model*, it will automatically use GRG2. GRG2 may have difficulty with some LP or QP problems that could have more easily been solved by another method. Beware the black box!

• ***Integer programs***: Branch and Bound

## Solution Method Intuition

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- All of the solution methods just mentioned tend to have very intuitive explanations.
- We will talk about how the simplex method works by graphically solving a small (two variable) linear program.
- Again, our problem:

$$\max_{x_A, x_B} \bar{\Pi} = 300x_A + 250x_B$$

$$\text{such that: } 2x_A + x_B \leq 40$$

$$x_A + 3x_B \leq 45$$

$$x_A \leq 12$$

$$x_A \geq 0$$

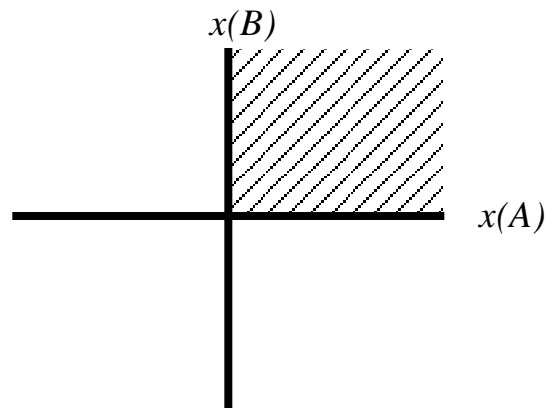
$$x_B \geq 0$$



## Graphical Solutions

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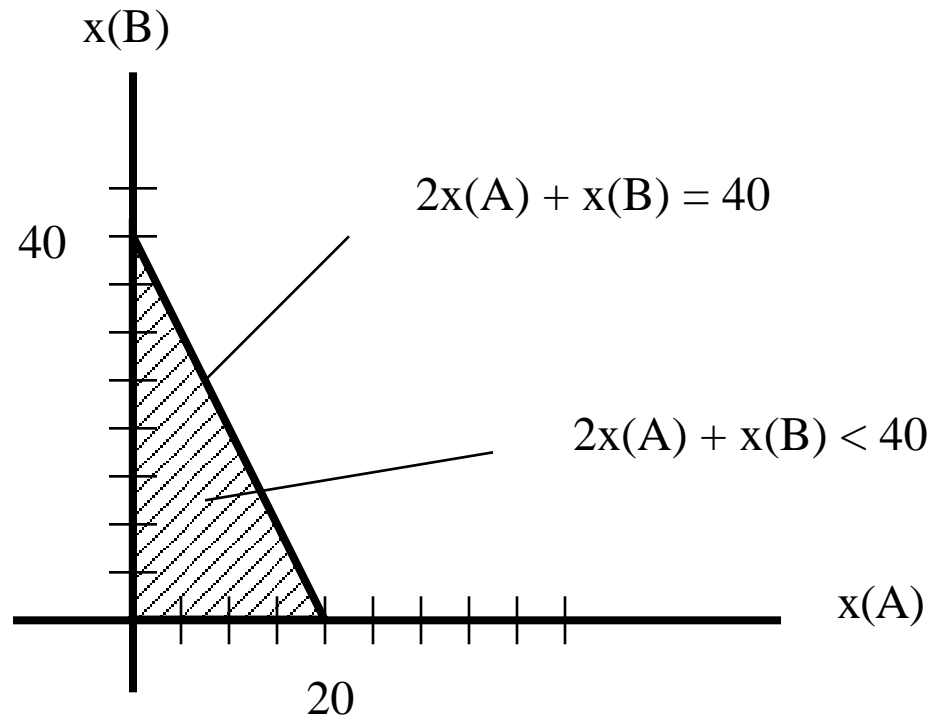
- Limited to either 2 constraints/variables.
- The objective function:  $\max_{x_A, x_B} \bar{\Pi} = 300x_A + 250x_B$
- The solution is contained *somewhere* in real-valued space. That's a big area. Fortunately, we can narrow it down immediately to the first quadrant because of the feasibility constraints:  $x_A \geq 0$  and  $x_B \geq 0$ .



## Step One: Graphing the Constraints

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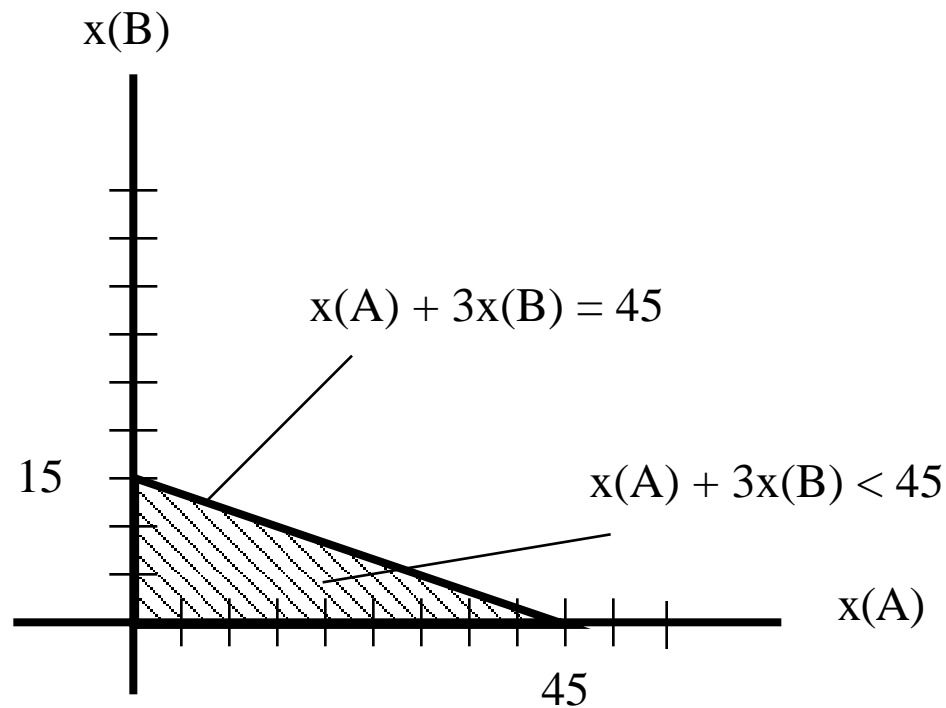
- Consider Constraint #1:  $2x_A + x_B \leq 40$
- When  $x_A = 0$ ,  $x_B \leq 40$ . To anchor the line, note that when  $x_B = 0$ ,  $x_A \leq 20$ .



## Constraint #2: $x_A + 3x_B \leq 45$

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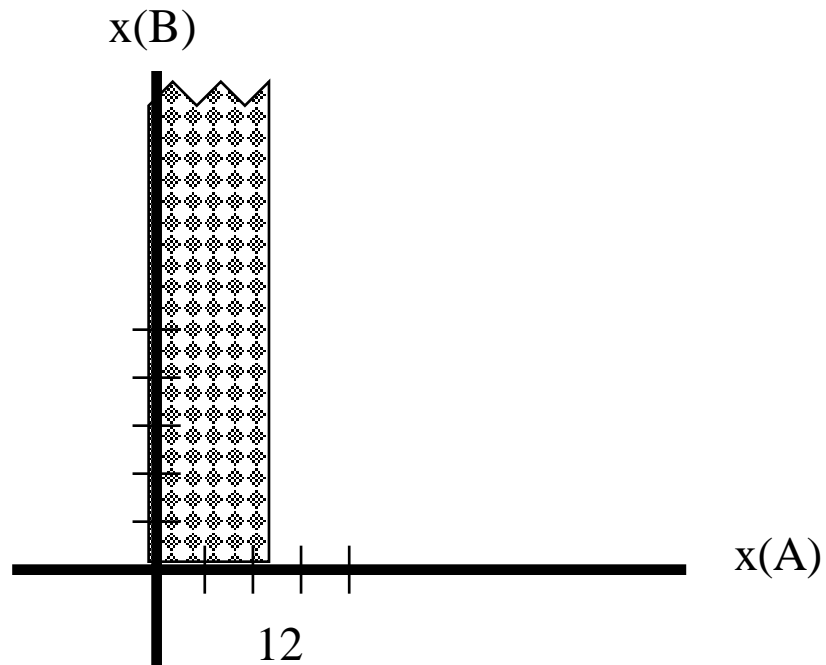
- When  $x_A = 0$ , then  $x_B \leq 15$ . Anchoring the line, when  $x_B = 0$ , then  $x_A \leq 45$ .



### Constraint #3: $x_A \leq 12$

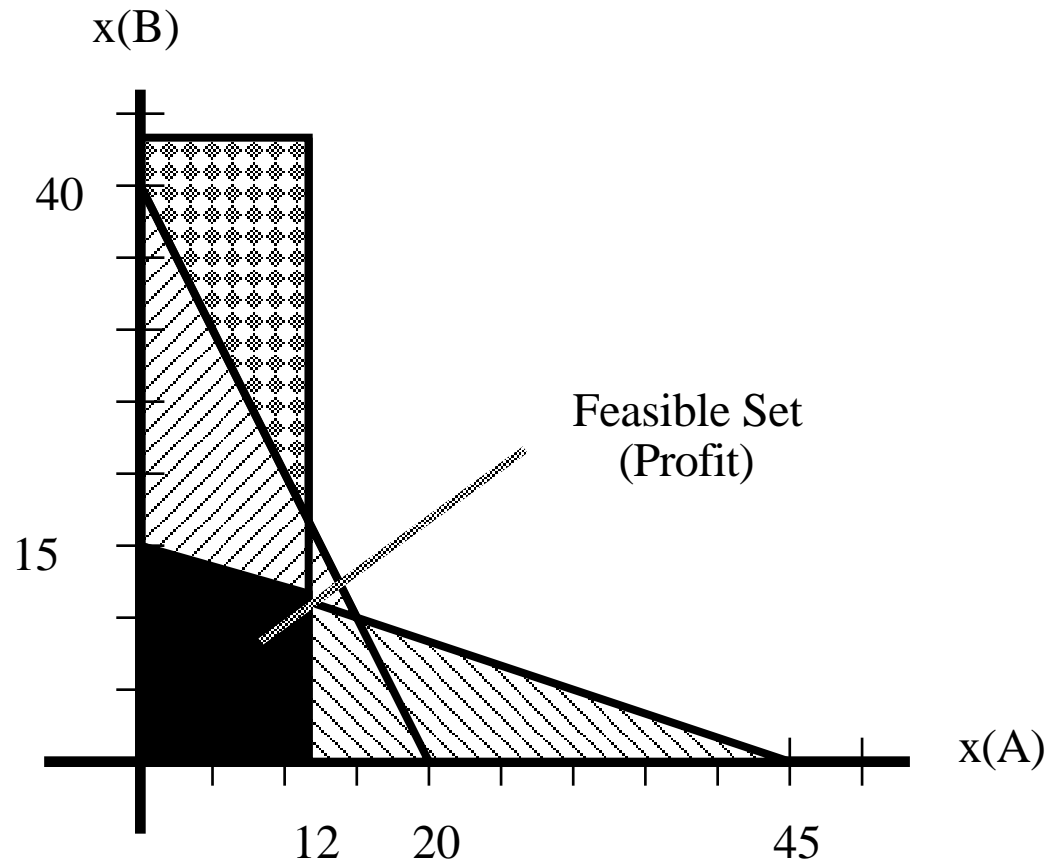
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- This constraint requires no solving for.



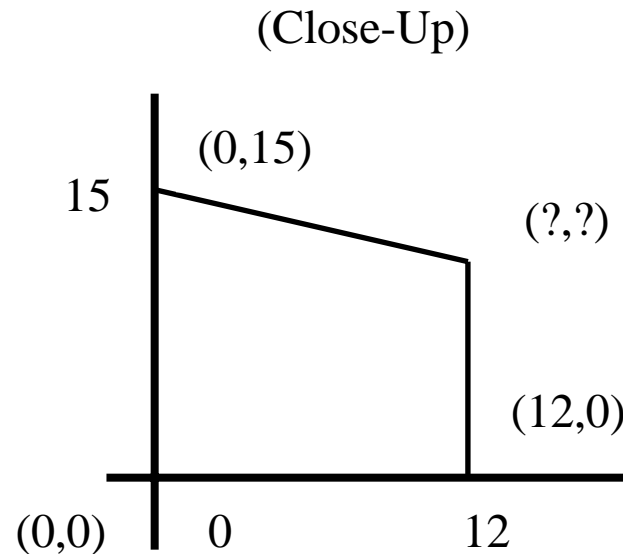
## Combining the Constraints: The Feasible Set

- The intersection of all constraint-feasible sets represents the set of feasible solutions (if a solution exists). Keep in mind that this is the *feasible* set - it says nothing about the *optimal* set!



## Identifying the Optimal Points Within the Feasible Set

- **Method #1:** Corner-Point Enumeration



- The “mystery” point is the intersection of the two lines. Two equations, two unknowns:  $x_A = 12$  and  $x_A + 3x_B = 45$ . Therefore  $12 + 3x_B = 45 \therefore x_B = 11$ .

- Thus, the coordinates of the fourth corner are (12, 11).

## Choosing a Corner-Point

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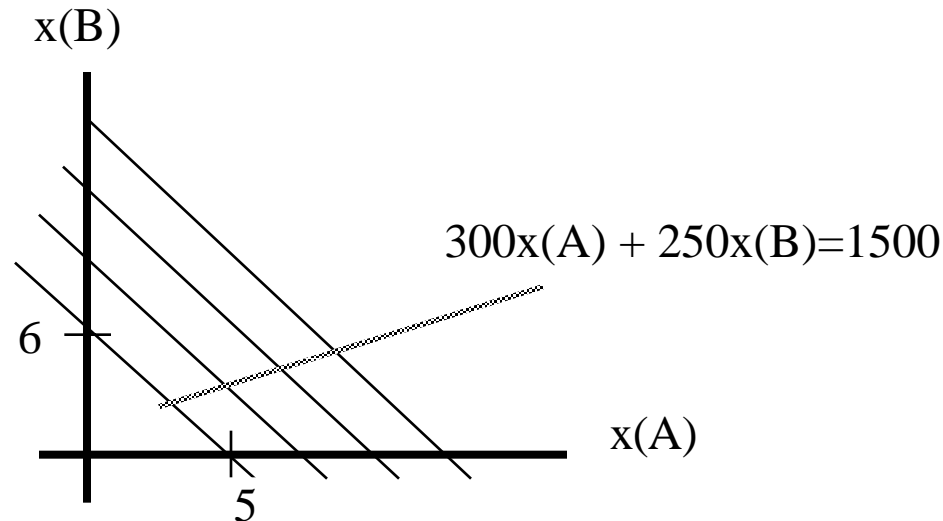
- Each of the corner-points represents a possible optimal solution. To identify the optimal one, substitute them into the objective function and see which one maximizes the function.

<b>Trial</b>	<b>Objective Function</b>	<b>Profit</b>
1	$300(0) + 250(0)$	\$-0-
2	$300(0) + 250(15)$	\$3,750
3	$300(12) + 250(0)$	\$3,600
★ 4 ★	$300(12) + 250(11)$	<b>\$6,350</b>

- But why limit ourselves to corner-points? Why shouldn't we choose a point in the interior of the feasible set?

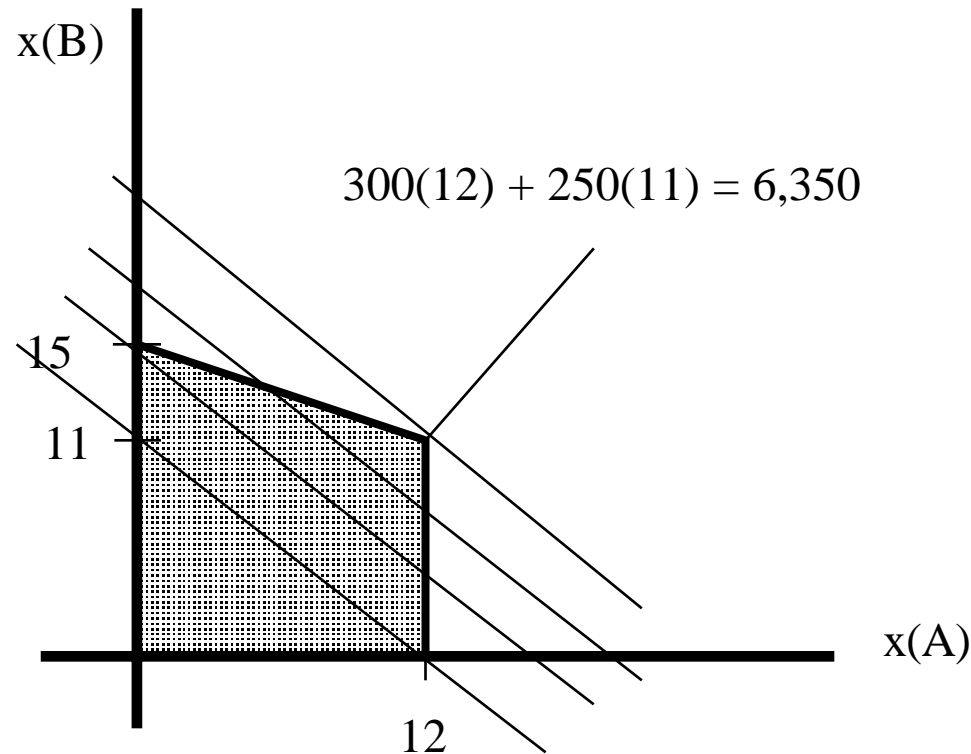
## Isoprofit (Isocost) Curves

- $x_A$  and  $x_B$  can take many values. To identify the optimal values, substitute arbitrary values in as the objective function and then change them to push the frontier out towards the boundary of the feasible set.
- To start, pick a value for the objective function - say, \$1,500. This becomes *an* isoprofit curve:  $300x_A + 250x_B = 1500$ .
- When  $x_A = 0$ ,  $x_B = 6$  and when  $x_B = 0$ ,  $x_A = 5$ .





## Adding Constraints to the Isoprofit Curves



- Any further advancement of the isoprofit curve would cause it to leave the feasible set. Thus, (1) optimal solutions exist only on boundaries, and (2) if only one exists, the optimal solution must always occur at a vertex.

## A Quick Review

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- LPs are good for optimization problems involving maximizing profits and minimizing costs.
- LP = Decision Variables, Objective Function, and Constraints
- Decision Variables** are the quantities of resource being allocated
- The **Objective Function** is what's being optimized.
- Constraints** are resource limitations or requirements
- Advantages of Linear Programming:
  - Relatively quick
  - Guaranteed to find optimal solution
  - Provides natural sensitivity analysis (shadow prices)

## **Disadvantages of Linear Programming**

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- Absence of risk (does expectation adequately capture risk?)
- Restriction to linear objective functions
  - No correlation among variables
  - No positive or negative synergies
  - Nonlinear programming is very difficult
- Fractional solutions often have no meaning
- Reducing the world to a set of linear equations is usually very difficult

## Solution Methods

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- The Simplex Method:

- (1) Pick a trial solution point
- (2) Move out from that point along the edges, looking for vertices.
- (3) Keep moving along the positive gradient until no further improvements can be made.

- The Graphical Analogue to the Simplex:

- (1) Draw the constraints
- (2) Push the isoquant as far in the optimal direction as possible.
- (3) Pick the last vertex exiting the feasible set.

- For more graphing help, see *The Geometry of Linear Programming* at <http://equilibrium.hss.cmu.edu/dadss/>